LA-UR-79-251

MASTER

CONF-790629--2

TITLE: LIE-GROUP THEORY FOR SYMBOLIC INTEGRATION OF FIRST-ORDER DIFFERENTIAL EQUATIONS: A MODERN TREATMENT

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SUBMITTED TO: MAXSYMA Users Conference Proceedings

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LIE-GROUP THEORY FOR SYMBOLIC INTEGRATION

OF FIRST-ORDER DIFFERENTIAL EQUATIONS:

A MODERN TREATMENT

by

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ABSTRACT

A review of the present status of the application of Lie-group theory to the solution of first order ordinary differential equations (ODE's) is given. A code written in the MACSYMA language is presented which finds and solves first order ODE's invariant under group with infinitesimal generation of the form $U = A(x)B(y)\partial_x + C(x)D(y)\partial y$. An algorithm is given by which one can begin with an ODE $y' = f(x,y)$ with known solution $\phi(x,y) = c$ and obtain a possibly larger class of ODE's with solutions given in closed form. A final algorithm for forming a sequence of solvable differential equations is suggested. The work can be generalized to higher order differential equations, partial differential equations, and difference equations.

"The problem of integrating a differential equation will be reduced to the problem of finding a one-parameter group which leaves the equation unaltered,... We do not attempt to integrate the equation ignoring the data of the problem in which it arose, but rely on the data to suggest a group which leaves the equation unaltered." -
L. E. Dickson [5, page 303].

CONTENTS

1. INTRODUCTION

This paper reviews the present status of the contributions which Lie group theory can make to the solution in finite terms of first order ordinary differential equations (ODE's):

$$y' = f(x,y) \tag{1.1}$$

The paper discusses four topics:

1. A review of the relevant Lie group theory. This is included since there is not an adequate summary in one place of exactly those parts of Lie group theory relevant to the above problem.
2. A code written in the MACSYMA language to find and solve equations (1.1) which are invariant under the Lie group with infinitesimal generator of the form

$$U = A(x)B(y)\partial_x + C(x)D(y)\partial_y \quad . \tag{1.2}$$

3. An algorithm by which one can begin with an equation (1.1) with a known one-parameter family of solutions

$$\phi(x,y) = C \tag{1.3}$$

and obtain, using Lie group methods, a possibly larger set of equations of the form (1.1) with known solution. Thereby, one can make use of tables such as Kamke [8] and Murphy [12] to extend the domain of equations (1.1) with solutions given in finite form. The theory of such a project is given, but is not implemented other than for two examples. This theory makes possible a more systematic organization and enlarging of tables of solved ODE's.

4. An algorithm is outlined by which one can start with an ODE with known solution and construct a sequence of first order ODE's with solution derivable from the previous members of the sequence.

(One would really like, for a given ODE, to find its invariance group and thereby solve the ODE. It turns out that this task is exactly equivalent to solving the ODE in closed form. However, Dickson [5] using examples of ODE's from geometry, illustrates how in certain situations one can guess the appropriate invariance group.)

By solving (1.1) in finite terms, we mean that f(x,y) is a given elementary function of two variables (built up from algebraic, exponential, and logarithmic operations) and one expresses the solution (1.3) in terms of functions of two variables, built up from a finite number of algebraic, exponential, and integral operations, when such a solution exists. One goal of those who work in this subject is to determine, for a given elementary f, if such a ϕ exists and to determine its value when it exists. As is well known, Risch ([14], [15]) has discovered and implemented such an algorithm, loosely speaking, when f is a function of a single variable.

Future projects include the implementation of item 3, above, the extension of this work to higher order ODE's, partial differential equations, and difference equations.

The material of Part I is taken from Kamke [7], Markus [11], and Pontryagin [13] and is included in this report to bring into one place exactly those parts of Lie group theory needed for integration of first order ODE's.

Part I. Theory

2. Local Lie Group

A topological space G is a local group if for some pairs a,b of points in G, ab is defined and the following conditions are satisfied. An upper case Latin letter denotes a neighborhood of its argument.

a. (ab)C = a(bc) if the four products stated are defined.

b. If ab is defined, then for every W(ab) there exists U(a) and V(b) so that $U(a)V(b) \subset W(ab)$ where U(a)V(b) is the collection of products xy, $x \in U$, $y \in V$.

c. There is an identity $e \in G$ so that ea is defined and ea = a for all $a \in G$.

d. If for $b \in G$ there is a left inverse $b_L^{-1} \in G$ so that $b_L^{-1} b = e$, then for every $U(b_L^{-1})$ there exists V(b) so that for $y \in V(b)$ there exists $y_L^{-1} \in U(b_L^{-1})$.

Lemma 1. For any local group G and for any W(e), there exists U(e) so that for $a,b \in V(e)$, $ab \in W(e)$.

Let the local group G have topological dimension r. We say that a coordinate system is defined in G if there is a homeomorphism H of U(e) onto $V(\underset{\sim}{0})$, where $\underset{\sim}{0}$ is the origin of r-dimension Euclidean space R^r, so that $H(e) = \underset{\sim}{0}$.

For W(e), $e \in G$, let U(e) be the neighborhood of e as defined in Lemma 1. For x, $y \in U(e)$, put z = xy. If $H(x) = \underset{\sim}{x}^r$, $H(y) = \underset{\sim}{y}^r$, and $H(z) = \underset{\sim}{z}^r$, put

$$\underset{\sim}{z}^r = \underset{\sim}{f}\,(\underset{\sim}{x}^r, \underset{\sim}{y}^r).$$

$\underset{\sim}{f}$ are said to be differentiable if it is three times differentiable.

Definition 1. A local group G is a local Lie group if it is possible to introduce differentiable coordinates into G. r is then called the number of parameters of G.

Definition 2. Let G be an r-parameter local Lie group and Γ and Δ be open sets in R with $\Delta \supset \Gamma$. Suppose for $x \in G$, $\phi_x: \Gamma \to \Delta$ is a homeomorphism so that $\phi_x(\xi)$ is a continuous function of x for each fixed $\xi \in \Gamma$. G is a local Lie group of transformations if:

a. For $e \in G$, $\phi_e(\xi) = \xi$ for all $\xi \in \Gamma$

b. For x,y sufficiently small in the topology of G, $\phi_x(\phi_y(\xi)) = \phi_{xy}(\xi)$.

c. $\phi_x \equiv \phi_e$ only if $x = e$.

d. $\phi_x(\xi)$, regarded as a map from $R^r \times R^n$ to R^n, is a three times differentiable function.

(The reason for the appearance of the number "three" is that "three" is number of derivatives which assures that other functions arising in most traditional computations with Lie groups will have the correct number of derivatives.)

3. First Order Ordinary Differential Equations

In all that follows, $\mathcal{O}$ will denote an open connected set in R^2. We need the following existence theorem for first order ordinary differential equations, taken from Arnold [1]. See pages 48 and 226 for what Arnold calls the rectification theorem (the hypotheses are rather scattered in the book). Consider

$$\dot{\underset{\sim}{x}} = \underset{\sim}{V}(\underset{\sim}{x}) \quad , \tag{3.1}$$

where $\underset{\sim}{x} \in R^2$ and $\underset{\sim}{V}: \mathcal{O} \to R^2$. Let $r \geq 2$ be some integer, including ∞. Suppose the components of $\underset{\sim}{V}$ are of class C^r in $\mathcal{O}$. Suppose $\underset{\sim}{x}_0 \in \mathcal{O}$ and $\underset{\sim}{V}(\underset{\sim}{x}_0) \neq \underset{\sim}{0}$. Then there exists a solution $\underset{\sim}{x} = \underset{\sim}{\phi}(t,\underset{\sim}{x}_0)$ satisfying $\underset{\sim}{\phi}(t_0,\underset{\sim}{x}_0) = \underset{\sim}{x}_0$ in some interval containing t_0. The function $\underset{\sim}{\phi}(t,\underset{\sim}{x}_0)$ in some neighborhood N of $(t_0,\underset{\sim}{x}_0)$ is in C^{r-1} and for N sufficiently small, the solution is unique.

One can remove the restriction that $\underset{\sim}{V}(\underset{\sim}{x}_0) \neq 0$ by combining theorems in Birkhoff and Rota [2, pages 151, 152, and 162]. Namely, if the components of $\underset{\sim}{V}(\underset{\sim}{x})$ are continuously differentiable in a bounded closed convex domain $\mathcal{O}$, then (3.1) has through each point $\underset{\sim}{x}_0$ of $\mathcal{O}$ a unique solution for some interval $|t-t_0| \leq T$ for some T possibly depending on $\underset{\sim}{x}_0$.

4. Local One-Parameter Transformation Groups on R^2

Theorem 1. Suppose $f_i(\underset{\sim}{x}) \in C^\infty(\mathcal{O})$ for $i = 1,2$ and $\underset{\sim}{f} = (f_1,f_2) \neq \underset{\sim}{0}$ in $\mathcal{O}$. Let

$$\mathcal{U} = \left\{ \frac{dx}{dt} = \underset{\sim}{f}(\underset{\sim}{x}) \right\} \tag{4.1}$$

define a vector field in $\mathcal{O}$. Let $\underset{\sim}{\phi}(t,\underset{\sim}{x}_0)$ be the unique solution curve of (4.1) through $\underset{\sim}{x}_0 \in \mathcal{O}$ at $t = 0$. Then $\underset{\sim}{\phi}(t,\underset{\sim}{x})$ defines a one-parameter local transformation group $\{T_t\}$ on $\mathcal{O}$ which is said to be generated by $\mathcal{U}$.

Proof. From Section 3, $\underset{\sim}{\phi}(t,\underset{\sim}{x}) \in C^\infty$ in $\bar{R}^1 \times \mathcal{O}$ where $\bar{R}^1$ is some interval of the t-axis containing t=0. For t sufficiently small in magnitude,

$$\frac{\partial}{\partial s} \underset{\sim}{\phi}(s + t, x_0) = \frac{d}{ds} \underset{\sim}{x}(s+t)$$

$$= \underset{\sim}{f}(\underset{\sim}{x}(s+t)) = \underset{\sim}{f}(\underset{\sim}{\phi}(s+t,\underset{\sim}{x}_0)).$$

Thus for each fixed sufficiently small t and for sufficiently small s, $\underset{\sim}{\phi}(s+t,\underset{\sim}{x}_0)$ is the unique solution of $\mathcal{U}$ through $\underset{\sim}{\phi}(t,\underset{\sim}{x}_0) = \underset{\sim}{x}_1$ at s=0. Thus

$$\underset{\sim}{\phi}(s+t,\underset{\sim}{x}_0) = \underset{\sim}{\phi}(s,\underset{\sim}{x}_1) \quad . \tag{4.2}$$

Now define the one-parameter transformation of $\mathcal{O}$ by

$$\underset{\sim}{x}_1 = \underset{\sim}{\phi}(t_1,\underset{\sim}{x}_0)$$

for t_1 sufficiently small. Put, for t_2 sufficiently small,

$$\underset{\sim}{x}_2 = \underset{\sim}{\phi}(t_2,\underset{\sim}{x}_1) \quad .$$

Then

$$\underset{\sim}{x}_2 = \underset{\sim}{\phi}(t_2,\underset{\sim}{\phi}(t_1,\underset{\sim}{x}_0)) \quad . \tag{4.3}$$

From (4.2) ,

$$\underset{\sim}{x}_2 = \underset{\sim}{\phi}(t_1 + t_2,\underset{\sim}{x}_0) \; . \tag{4.4}$$

Comparing (4.3) and (4.4), we see that $\underset{\sim}{\phi}(t,\underset{\sim}{x})$ defines a local one-parameter transformation group.

Definition 3. The vector field (4.1) with $\underset{\sim}{f} \in C^{\infty}(\mathcal{O})$ and $\underset{\sim}{f} \neq 0$ in $\mathcal{O}$ is called an infinitesimal one-parameter transformation group on $\mathcal{O}$.

Theorem 2. Each local one-parameter transformation group on $\mathcal{O}$ is generated by one and only one infinitesimal one-parameter transformation group.

Proof. Let $\underset{\sim}{\phi}(t,\underset{\sim}{x})$ be a local one-parameter transformation group on $\mathcal{O}$. Define the infinitesimal one-parameter group $\mathcal{U}$ (see (4.1)) by

$$\left.\frac{\partial \underset{\sim}{\phi}}{\partial t}(t,\underset{\sim}{x})\right|_{t=0} = \underset{\sim}{f}(\underset{\sim}{x}) \quad .$$

Let $\underset{\sim}{\phi}^{1}(t,x)$ be generated by U as in Theorem 1. $\underset{\sim}{\phi}$ and $\underset{\sim}{\phi}^{1}$ have the same initial value. We now show that $\underset{\sim}{\phi}$ and $\underset{\sim}{\phi}^{1}$ satisfy the same system of differential equations. By the group property, when s and t are sufficiently small,

$$\underset{\sim}{\phi}(s+t,\underset{\sim}{x}) = \underset{\sim}{\phi}(s,\underset{\sim}{\phi}(t,\underset{\sim}{x}))$$

so

$$\underset{\sim}{\phi}_t(s+t,\underset{\sim}{x}) = \underset{\sim}{\phi}_t(s,\underset{\sim}{\phi}(t,\underset{\sim}{x})) \quad .$$

Put s = 0:

$$\underset{\sim}{\phi}_t(t,\underset{\sim}{x}) = \underset{\sim}{\phi}_t(0,\underset{\sim}{\phi}(t,\underset{\sim}{x})) \quad .$$

Thus

$$\underset{\sim}{\phi}_t(t,\underset{\sim}{x}) = \underset{\sim}{f}(\phi(t,\underset{\sim}{x})) \quad .$$

Thus $\underset{\sim}{\phi}$ and $\underset{\sim}{\phi}^{1}$ satisfy the same ODE and have the same initial value and therefore in $\mathcal{O}$ are equal. Conversely, distinct vector fields have distinct integral curves.

Theorem 3. Let $h(\underset{\sim}{x}): \mathcal{O} \to R^{1}$ be real analytic (C^{ω}) and let

$$U = \underset{\sim}{f}(x) \cdot \underset{\sim}{\partial} \quad , \tag{4.8}$$

where $\underset{\sim}{\partial} = (\partial/\partial x_1, \partial/\partial x_2)$, be a differential operator with $\underset{\sim}{f} \subset C^{\omega}(\mathcal{O})$.

Define

$$\hat{h}(t,\underset{\sim}{x}) = h(\underset{\sim}{\phi}(t,\underset{\sim}{x})) \quad , \tag{4.9}$$

where $\underset{\sim}{\phi}(t,\underset{\sim}{x})$ is defined by the vector field (4.1) in $\mathcal{O}$ and t is sufficiently small in magnitude. Then

$$\hat{h}(t,\underset{\sim}{x}) = e^{tU}\, h(\underset{\sim}{x}) \quad ,$$

where e^{tU} is the power series of operators:

$$e^{tU} = \sum_{n=0}^{\infty} \frac{t^n}{n!} U^n \quad .$$

Proof. We first use an induction argument to establish that

$$U^n\, h(\underset{\sim}{x}) = \left. \frac{\partial^n \hat{h}(t,\underset{\sim}{x})}{\partial t^n} \right|_{t=0} \quad . \tag{4.10}$$

For n = 1,

$$\left. \frac{\partial \hat{h}(t,\underset{\sim}{x})}{\partial t} \right|_{t=0} = \left. \frac{\partial}{\partial t}\, h(\underset{\sim}{\phi}(t,\underset{\sim}{x})) \right|_{t=0} \tag{4.11}$$

$$= h_{\underset{\sim}{x}} \cdot \underset{\sim}{f} = Uh(\underset{\sim}{x}) \quad .$$

Assume

$$\left. \frac{\partial^{n-1} \hat{h}(t,\underset{\sim}{x})}{\partial t^{n-1}} \right|_{t=0} = U^{n-1} h(\underset{\sim}{x}) \quad . \tag{4.12}$$

Then

$$U\left[U^{n-1} h(\underset{\sim}{x})\right] = \left. \frac{\partial}{\partial s}\, \hat{U}^{n-1}(s,\underset{\sim}{x}) \right|_{s=0} \qquad \text{(by (4.12))}$$

$$= \left. \frac{\partial}{\partial s}\, U^{n-1} h\left(\underset{\sim}{\phi}(s,\underset{\sim}{x})\right) \right|_{s=0} \qquad \text{(by (4.10))}$$

Definition 4. If $\mathcal{U}$ is the vector field (4.1) with $\underset{\sim}{f} \in C^{\infty}(\mathcal{O})$, the operator U (4.8) is called the (infinitesimal) generator of the local one-parameter transformation group on $\mathcal{O}$ given in Theorem 2.

Definition 5. Let U be the generator of a local one-parameter transformation group $\{T_t\}$ in $\mathcal{O}$. A function $h: x \to R^1$ with $h \in C^{\infty}(\mathcal{O})$ is invariant under U if

$$h(T_t \underset{\sim}{x}) = h(\underset{\sim}{x})$$

for all $\underset{\sim}{x} \in \mathcal{O}$.

Theorem 4. $h(\underset{\sim}{x}) \in C^{\infty}(\mathcal{O})$ is invariant under U if and only if

$$Uh \equiv 0$$

in $\mathcal{O}$.

Proof. Suppose h invariant. Then

$$\hat{h}(t,\underset{\sim}{x}) = h\left(\underset{\sim}{\phi}(t,\underset{\sim}{x})\right)$$

is independent of t:

$$\frac{\partial}{\partial t}\hat{h}(t,\underset{\sim}{x}) = 0 \ .$$

But

$$\frac{\partial}{\partial t}\hat{h}\Bigg|_{t=0} = Uh \quad .$$

Thus $Uh = 0$ for each $\underset{\sim}{x} \in \mathcal{O}$.

Conversely, suppose $Uh \equiv 0$ in $\mathcal{O}$. Then

$$= \frac{\partial}{\partial s} \frac{\partial^{n-1}}{\partial t^{n-1}} \hat{h}\left(t, \underset{\sim}{\phi}(s, \underset{\sim}{x})\right)\Bigg|_{\substack{t=0 \\ s=0}} \qquad \text{(by (4.12))}$$

$$= \frac{\partial}{\partial s} \frac{\partial^{n-1}}{\partial t^{n-1}} h\left(\underset{\sim}{\phi}(t, \underset{\sim}{\phi}(s, \underset{\sim}{x}_1))\right)\Bigg|_{\substack{t=0 \\ s=0}} \qquad \text{(by (4.9))}$$

$$= \frac{\partial}{\partial s} \frac{\partial^{n-1}}{\partial t^{n-1}} h\left(\underset{\sim}{\phi}(t+s, x)\right)\Bigg|_{\substack{t=0 \\ s=0}}$$

(using the group property of $\underset{\sim}{\phi}$)

$$= \frac{\partial}{\partial s} \frac{\partial^{n-1}}{\partial t^{n-1}} \hat{h}\ (t+s, \underset{\sim}{x})\Bigg|_{\substack{t=0 \\ s=0}} \qquad \text{(by (4.9))}$$

$$= \frac{\partial^n}{\partial t^n} \hat{h}\ (t, \underset{\sim}{x})\Bigg|_{t=0} .$$

This establishes (4.10).

Since $\underset{\sim}{\phi}(t,\underset{\sim}{x})$ is analytic in t for each $\underset{\sim}{x}$ in $\mathcal{O}$ for sufficiently small t (see Lefschetz [9], page 14), $\hat{h}$ is analytic in t and therefore,

$$\hat{h}(t,\underset{\sim}{x}) = h(\underset{\sim}{x}) + tUh + \frac{t^2}{2!} U^2 h + \frac{t^3}{3!} U^3 h + \ldots,$$

which is the conclusion of the theorem.

Corollary 1. For $\underset{\sim}{f} \in C^{\omega}(\mathcal{O})$ and for t small,

$$\phi_i(t,\underset{\sim}{x}) = e^{tU} x_i \quad , \quad i = 1,2.$$

$$\hat{h}(t+s,\underset{\sim}{x}_0) = h\left(\underset{\sim}{\phi}(t+s,\underset{\sim}{x}_0)\right)$$

and

$$\hat{h}(t+s,\underset{\sim}{x}_0) = h\left(\underset{\sim}{\phi}(s,\underset{\sim}{x}_1)\right) \quad ,$$

where

$$\underset{\sim}{x}_1 = \underset{\sim}{\phi}(t,\underset{\sim}{x}_0) \quad .$$

Then

$$\left.\frac{\partial \hat{h}}{\partial s}(t+s,x_0)\right|_{s=0} = \left.Uh\right|_{\underset{\sim}{x}=\underset{\sim}{x}_1} = 0$$

by previous work. Thus

$$\frac{\partial \hat{h}}{\partial t}(t,\underset{\sim}{x}_0) = 0$$

for each t, where defined.

We now use the notation $\underset{\sim}{x} = (x,y)$.

Definition 6. A differential equation

$$D: \frac{dy}{dx} = f(x,y) \quad ,$$

where $f \in C^\infty(\mathcal{O})$, is invariant under a local one-parameter Lie group $\{T_t\}$ on $\mathcal{O}$ if $(x,y) \in \mathcal{O}$ and $(x_1,y_1) \in \mathcal{O}$ is on the unique solution to D through (x,y) then the same holds for $(T_t x, T_t y)$ and $(T_t x_1, T_t y_1)$.

Definition 7. For a given open connected set $\mathcal{O}\subset R^2$, let $L(\mathcal{O})$ be the Cartesian product of $\mathcal{O}$ and S where S is the circle with the natural topology of a circle, the circle being the one point compactification of the line.

Lemma 2. A differential equation $Mdx+Ndy=0$, $M^2+N^2>0$ in $\mathcal{O}$, $M,N \in C^{\infty}(\mathcal{O})$ defines a differential surface in $L(\mathcal{O})$ above $\mathcal{O}$ where a point of the surface is given for $(x,y) \in \mathcal{O}$ by the triplet $(x,y,\ dy/dx = -M(x,y)/N(x,y))$ where if $N=0$, dy/dx is associated with the exceptional point of the circle.

Definition 8. A mapping $f:\mathcal{O}_1 \to \mathcal{O}_2$, where $\mathcal{O}_i$ are open connected sets in R^2, is a diffeomorphism if f is one-to-one onto and f and $f^{-1}\in C^1$.

Lemma 3. A diffeomorphism $U = U(x,y)$, $V = V(x,y)$ between open sets $\mathcal{O}_1,\mathcal{O}_2\subset\mathcal{O}$ defines a diffeomorphism between $L(\mathcal{O}_1)$ and $L(\mathcal{O}_2)$ by the mapping

$$(x,y,p) \to (U(x,y),V(x,y),q) \tag{4.13}$$

where

$$q = \frac{dV}{dU} = \frac{dV(x,y(x))/dx}{dU(x,y(x))/dx} = \frac{\partial V/\partial x+p\partial V/\partial y}{\partial U/\partial x+p\partial U/\partial y} \quad . \tag{4.14}$$

Corollary 2. A local transformation group $\underset{\sim}{\phi}(t,\underset{\sim}{x}) = (\phi(t,x,y),\ \psi(t,x,y))$ in $\mathcal{O}$ defines a local transformation group in $L(\mathcal{O})$ by

$$(x,y,p) \to(\phi(t,x,y),\ \psi(t,x,y),\ X(t,x,y,p)) \quad , \tag{4.15}$$

where

$$X(t,x,y,p) = \frac{d\psi}{d\phi} = \frac{d\psi/dx}{d\phi/dx} = \frac{\frac{\partial\psi}{\partial x}(t,x,y)+p\frac{\partial\psi}{\partial y}(t,x,y)}{\frac{\partial\phi}{\partial x}(t,x,y) + p\frac{\partial\phi}{\partial y}(t,x,y)} \quad . \tag{4.16}$$

5. Extended Transformation Group

Theorem 5. Let $U = f(x,y)\partial_x + g(x,y)\partial_y$ be the generator of a local one-parameter group G on $\mathcal{O}$. The generator of the associated local one-parameter transformation group in $L(\mathcal{O})$ is

$$U' = f(x,y)\partial_x + g(x,y)\partial_y + h(x,y,p)\partial_p \tag{5.1}$$

where

$$h(x,y,p) = g_x + (g_y - f_x)p - f_y p^2 \quad . \tag{5.2}$$

U' is called the generator of the once-extended transformation group G^1.

Proof. Expressions (4.15) and (4.16) define the group. To find its generator, use the construction given in Theorem 2, when extended to three variables. Thus

$$\left.\frac{\partial\phi}{\partial t}\right|_{t=0} = f(x,y) \quad , \quad \left.\frac{\partial\psi}{\partial t}\right|_{t=0} = g(x,y) \tag{5.3}$$

and

$$\left.\frac{\partial \chi}{\partial t}\right|_{t=0} = \left.\frac{\left(\phi_x+\phi_y p\right)\left(\psi_{xt}+\psi_{yt}p\right)-\left(\psi_x+\psi_y p\right)\left(\phi_{xt}+\phi_{yt}p\right)}{\left(\phi_x+p\phi_y\right)^2}\right|_{t=0} . \tag{5.4}$$

Using $\phi = e^{tU}x$, $\psi = e^{tU}y$, one obtains

$$\phi_x\Big|_{t=0} = 1 \quad , \quad \phi_y\Big|_{t=0} = 0 \tag{5.5}$$

and

$$\psi_x\Big|_{t=0} = 0 \quad , \quad \psi_y\Big|_{t=0} = 1 \quad . \tag{5.6}$$

Also,

$$\phi_{xt}\Big|_{t=0} = f_x \quad , \quad \phi_{yt}\Big|_{t=0} = f_y \tag{5.7}$$

and

$$\psi_{xt}\Big|_{t=0} = g_x \quad , \quad \psi_{yt}\Big|_{t=0} = g_y \quad . \tag{5.8}$$

Putting (5.5) through (5.8) into (5.4) and using (5.3), one obtains (5.1) and (5.2).

Definition 9. A surface W in $L(\mathcal{O})$ is invariant under a transformation T of $L(\mathcal{O})$ if $T(W) \subset W$.

Theorem 6. A differential equation D:

$$y' + M(x,y)/N(x,y) = 0 \qquad \left(M^2 + N^2 > 0\ ,\ M,\ N \in C^{\infty}(\mathcal{O})\right)$$

is invariant under the local transformation group generated by $U = f(x,y)\partial_x + g(x,y)\partial_y$ $(f,g \in C^{w}(\mathcal{O}))$ (see Definition 7) if and only if the surface defined by D in $L(\mathcal{O})$ is invariant under the local group generated by U' (see Definition 9).

Proof. The surface defined by D is invariant under the local group generated by U' if and only if the trajectories of U' remain in the surface and this is so if and only if the vector (f,g,h) is tangent to the surface. (f,g,h) is tangent to the surface $p=w(x,y)$ if and only if

$$f(-w_x) + g(-w_y) + h(1) = 0$$

or

$$U'(p-w) = 0 \cdot \tag{5.9}$$

on the surface of D.

6. Integrating Factor

Theorem 7. Suppose D: $M dx + N dy = 0$ $(M^2 + N^2 > 0$ in $\mathcal{O})$ is local group invariant under the local group generated by $U = f\partial_x + g\partial_y$, $fM + gN \neq 0$ in $\mathcal{O}$ and $f,M,g,N \in C^{\infty}(\mathcal{O})$. Then $\mu = 1/(fM + gN)$ is an integrating factor for D so that $(\mu M)_y = (\mu N)_x$.

Proof. Choose $P \in \mathcal{O}$ and assume $N(P) \neq 0$; otherwise $M \neq 0$ and interchange x and y. Now $U'\left(p + \frac{M}{N}\right) = 0$ on D in the subset of $L(\mathcal{O})$ lying above a neighborhood of P. Thus

$$\left[f\partial_x + g\partial_y + (g_x + (g_y - f_x)p - f_y p^2)\ \partial_p\right]\left[p + \frac{M}{N}\right] = 0$$

or

$$f\left(\frac{M}{N}\right)_x + g\left(\frac{M}{N}\right)_y + g_x + (g_y - f_x)\ p - f_y p^2 = 0\ .$$

But

$$p = -M/N \quad .$$

Thus

$$f\left(\frac{M}{N}\right)_x + g\left(\frac{M}{N}\right)_y + g_x\ (g_y - f_x)\ \left(-\frac{M}{N}\right) - f_y\left(\frac{M^2}{N^2}\right) = 0$$

which reduces finally to

$$\left(\frac{M}{fM+gN}\right)_y = \left(\frac{N}{fM+gN}\right)_x \quad .$$

7. Independent Functions

Definition 10 (Kamke [7, pages 302-3]). N functions $U_i(\underset{\sim}{x})$, $i = 1,2,\ldots,N$, defined with continuous first order partial derivatives on a closed bounded domain $B \subset R^N$, are dependent if there exists a function F: $R^N \to R^1$ so that the following hold:

a) F is defined on R^N with continuous first order partial derivatives;

b) F vanishes in no subdomain of R^N;

c) in B, $F(U_1(\underset{\sim}{x}),\ U_2(\underset{\sim}{x}),\ldots,U_N(\underset{\sim}{x})) = 0$

Definition 11. (Kamke [7, page 303]). N functions $U_i(\underset{\sim}{x})$: $R^N \to R^1$ are dependent in an open domain $\mathcal{O} \subset R^2$ if each in bounded closed subdomain $B \subset \mathcal{O}$ they are dependent by Definition 10. Else the U_i are independent.

8. Solution to $U'w=0$

Definition 12 (Kamke [7, page 323]). A set of integrals (solutions) $\psi_1,\ldots,\psi_{n-1}$ in a domain $G \subset R^n$ of the differential equation

$$\sum_{i=1}^{n} f_i\ (\underset{\sim}{x})\ \frac{\partial z}{\partial x_i} = 0 \qquad (8.1)$$

is called a principal system of integrals if the matrix:

$$\left(\frac{\partial\psi_i}{\partial x_j}\right)_{1 \le i \le n-1,\ 1 \le j \le N}$$

where $x = (x_1,x_2,\ldots,x_n)$, has rank n-1 in each subdomain of G.

Theorem 8. (Kamke [7, page 323]). Suppose the functions $f_i(\underset{\sim}{x})$, $1 \leq i \leq n$, are continuous in a domain $G \subset R^M$. If $\psi_i(\underset{\sim}{x})$, $1 \leq i \leq n-1$, are a principal system of integrals of (8.1), then a function $\psi(\underset{\sim}{x})$ is an integral of (8.1) if and only ψ and ψ_i are dependent in G (Definition 11).

Proof. See Kamke [7].

Corollary 3. Let u and v be independent solutions of the partial differential equation $U'w = 0$ in a domain $\mathcal{O} \subset R^2$, where U' is given by (5.1) and (5.2). Suppose $f,g \in C^\infty(\mathcal{O})$. An ODE $\ell(x,y,y') = 0$ of first order is invariant under the group generated by the infinitesimal transformation $U = f\partial_x + g\partial_y$ if and only if for each closed bounded subdomain $B \subset \mathcal{O}$ there exists a function F defined on R^3, has continuous first order partial derivatives, vanishes on no subdomain of R^3, and in B, $F(\ell,u,v) = 0$.

Part II. Applications

9. Algorithm 1.

We suppose that $f(x,y)$, $g(x,y) \in C^{\infty}(\mathcal{O})$ and therefore satisfy a Lipschitz condition in a convex and compact subdomain $U \subset \mathcal{O}$ (Arnold [1, page 218]). Two independent solutions to the equation

$$U' w = 0 \tag{9.1}$$

can be obtained by solving the associated characteristic equations:

$$\frac{dx}{dt} = f(x,y) \quad , \tag{9.2}$$

$$\frac{dy}{dt} = g(x,y) \quad , \tag{9.3}$$

and

$$\frac{dp}{dt} = h(x,y,p) \quad . \tag{9.4}$$

See Kamke [7, page 321]. By eliminating t from (9.2) and (9.3), one obtains

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \quad . \tag{9.5}$$

Let $(x_0,y_0) \in \mathcal{O}$ for which $f \neq 0$ or else $g \neq 0$. Then in a sufficiently small neighborhood N of (x_0,y_0) there is a one-parameter family of solutions of (9.5),

$$y = \omega(x,c) \tag{9.6}$$

where $y_0 = \omega(x_0,0)$, $f(x_0,y_0) \neq 0$ and $\omega(x,c) \in C^{\infty}$ near $(x_0,0)$ or

$$x = \theta(y,c) \tag{9.7}$$

where $x_0 = \theta(y_0,0)$, $g(x_0,y_0) \neq 0$, and $\theta(y,c) \in C^{\infty}$ near $(y_0,0)$. Then solving (9.6) or (9.7) for $c = u(x,y)$, $u(x,y)$ is a solution to (9.1) in the neighborhood N (Theorem 1 of Kamke [7, page 321]). A second solution, independent of u (Definition 11), is obtained as follows. If (9.6) is used, combine (9.4) with (9.2), else combine (9.4) with (9.3). If (9.6) is used, then (9.4) and (9.2) give

$$\frac{dp}{dx} = \frac{h(x,y,p)}{f(x,y)} \quad . \tag{9.8}$$

Using (9.6) in (9.8), gives

$$\frac{dp}{dx} = \frac{h(x,\omega(x,c),p)}{f(x,\omega(x,e))} \quad . \tag{9.9}$$

Solving (9.9) in a sufficiently small neighborhood of (x_0,p) for any p, one has a one-parameter family of solutions:

$$\rho(x,p,c) = \gamma \quad . \tag{9.10}$$

Then

$$v(x,y,p) = \rho(x,p,\omega(x,y)) \tag{9.11}$$

is a solution to (9.1). Similar calculations hold if (9.7) had been used in place of (9.6).

We now show that (9.9) is a Ricatti equation of the type that can be solved by integration. By (9.2), eq. (9.9) has the form

$$\frac{dp}{dx} = \frac{1}{f(x,w(x,c))} \left\{ g_x(x,w(x,c)) + \left[g_y(x,w(x,c) - f_x(x,w(x,c)) \right] p - f_y(x,w(x,c))\, p^2 \right\} . \tag{9.12}$$

The quantity

$$p = \frac{g(x,\omega(x,c))}{f(x,\omega(x,c))} \tag{9.13}$$

satisfies (9.12) as can be verified by substitution. Put

$$p = \frac{g(x,\omega(x,c))}{f(x,\omega(x,c))} + \frac{1}{H(x)} \quad . \tag{9.14}$$

as the general solution to (9.12). Then H(x) satisfies

$$\frac{dH}{dx} + \frac{1}{f} \left[g_y - f_x - \frac{2gf_y}{f} \right] H = \frac{f_y}{f} \quad . \tag{9.15}$$

On solving (9.15) for H and substituting into (9.14) we finally obtain the general solution to (9.12):

$$p = \frac{g(x,\omega(x,c))}{f(x,\omega(x,c))} + \frac{e^{\int \frac{g_y - f_x - 2gf_y/f}{f(x,\omega(x,c))}dx}}{\int \frac{f_y}{f} e^{\int \frac{g_y - f_x - 2gf_y/f}{f(x,\omega(x,c))}dx} dx + \gamma} \tag{9.16}$$

where γ is an arbitrary constant including ∞. The constant c in (9.16) is then replaced by u(x,y). On solving (9.16) for γ, one obtains

$$\gamma = \frac{1}{p - \frac{g(x,y)}{f(x,y)}} e^{\int (g_y - f_x - 2gf_y/f)/f \, dx}\Big|_{c \to u(x,y)}$$

$$- \int \frac{f_y}{f} e^{\int (g_y - f_x - 2gf_y/f)/f \, dx} dx \Big|_{c \to u(x,y)} . \tag{9.17}$$

Then if v(x,y,p) is set equal to the right hand side of (9.17), one obtains a second solution to (9.1) in a neighborhood of (x_0,y_0) if $f(x_0,y_0) \neq 0$.

If $f(x_0,y_0) = 0$, then $g(x_0,y_0) \neq 0$, and an appropriate formula to replace (9.17) can be obtained by use of (9.7).

In summary, one can start with any f and $g \in C^\infty(\mathcal{O})$. Obtain u and v. Then the ODE

$$F(u,v) = 0 \quad , \tag{9.18}$$

where F is described in Definition 10 when restricted to two variables, is invariant under the local group generated by $U = f\partial_x + g\partial_y$. (9.18) is solvable by use of the integrating factor given Theorem 7 provided (9.18) can be written in the form $y' = - M/N$.

If one selects one of the branches of (9.18) for v as a function of u: $v = W(u)$, then (9.18) gives:

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} + \frac{e^{\int (g_y - f_x - 2gf_y/f)/f\,dx}\Big|_{c \to w(x,y)}}{\int \frac{f_y}{f}\, e^{\int (g_y - f_x - 2gf_y/f)/f\,dx}\,dx\Big|_{c \to w(x,y)} + W(w(x,y))} \quad (9.19)$$

We include, as a special case of (9.19), the case of $\omega \equiv \infty$, so that

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \quad .$$

10. Application of Algorithm 1

Suppose

$$U = A(x)B(y)\partial_x + C(x)D(y)\partial_y \quad (10.1)$$

with $A,B,C,D \in C^{\infty}(e,f)$ for some interval (e,f). Equation (9.5) becomes

$$\frac{dy}{dx} = \frac{C(x)\ D(y)}{A(x)\ B(y)} \quad .$$

Suppose (a,b) and (c,d) are such that $A(x) \neq 0$, $B(y) \neq 0$ for $x \in (a,b)$ and $y \in (c,d)$. Put

$$u(x,y) = \int \frac{C(x)}{A(x)}\,dx - \int \frac{B(y)}{D(y)}\,dy \quad . \quad (10.2)$$

where the integrals are interpreted as indefinite integrals. Then $w = u(x,y)$ is a solution to (9.1). Put

$$u(x,y) = c \quad (10.3)$$

and select one of the branches resulting from solving (10.3) for y:

$$y = \omega(x,c) \quad .$$

Put

$$K(x,c) = \frac{C(x)D'(\omega(x,c))-A'(x)B(\omega(x,c))-2C(x)D(\omega(x,c))B'(\omega(x,c)/B}{A(x)B(\omega(x,c)}$$

$$= \frac{C(x)D'(\omega(x,c))}{A(x)B(\omega(x,c))} - \frac{A'(x)}{A(x)} - \frac{2C(x)D(\omega(x,c))}{A(x)B^2(\omega(x,c))} B'(\omega(x,c)) .$$

Then

$$e^{\int\frac{CD'}{AB}} = e^{\int\frac{dy}{dx}\frac{1}{D}\frac{dD}{dy}dx} = e^{\ell n\ D} = D ,$$

$$e^{-\int\frac{A'}{A}dx} = \frac{1}{A} ,$$

and

$$e^{-2\int\frac{C\ D\ B'}{A\ B^2}dx} = e^{-2\int\frac{dy}{dx}\frac{1}{B}\frac{dB}{dy}dx} = B^{-2} .$$

So

$$e^{\int K(x,c)dx} = \frac{D}{AB^2} .$$

So from (9.17),

$$v(x,y,p) = \frac{1}{p-\frac{C(x)D(y)}{A(x)B(y)}} \frac{D}{AB^2}$$

$$- \int\frac{B'(\omega(x,c))}{B^3(\omega(x,c))} \frac{D(\omega(x,c))}{A(x)} dx ,$$

or

$$v(x,y,p) = \frac{D(y)/A(x)B^2(y)}{p - \frac{C(x)D(y)}{A(x)B(y)}} - \int \frac{B'((\omega(x,c)D(\omega(x,c))}{A(x)B^3(\omega(x,c))}\,dx\,. \tag{10.4}$$

Thus a general first order ODE invariant under the group generated by (10.1) is given by

$$\frac{dy}{dx} = \frac{C(x)D(y)}{A(x)B(y)}$$

$$+ \frac{D}{AB^2}\left[\int \frac{B'(\omega(x,c)D(\omega(x,c))}{A(x)B^3(\omega(x,c))}\,dx\Bigg|_{c=u(x,y)} + W(u(x,y))\right]^{-1} \tag{10.5}$$

where W( ) is an arbitrary continuous function of one variable and u(x,y) is given by (10.2).

Eq. (10.5) is general enough to include the short lists of groups for first order ODE's given in Cohen [4], Dickson [5], Blumen and Cole [3], and Franklin [6], with the exception of the last group in Franklin's list.

11. MAXSYMA Code for $U = A(x)B(y)\partial_x + C(x)D(y)\partial_y$

MACSYMA is a general symbolic manipulation computer system in operation at the Laboratory for Computer Science of Massachusetts Institute of Technology. The MACSYMA system is described in the MACSYMA Reference Manual [10].

Table 1 gives a code (as described in Section 9) for obtaining a general first order ODE invariant under the group generated by $U = A(x)B(y)\partial_x + C(x)P(y)\partial_y$. The code is read as follows

On line 0 appears the title of the code: "Lie". The functions A,B,C,D, and W are read in. These functions appear in (10.5). The output of the code, or value of the function called "Lie", are three things: (1) The solution u(x,y) of (9.5) and denoted by B9 in the code.

(2) The right hand side of (10.5), denoted by B8 in the code. (3) The solution to the ODE (10.5) for the specified function W, the solution being denoted by F5 in the code. The code makes use of an "integrate" code which in turn makes use of parts of the Risch algorithm ([14], [15]).

In line 0, the symbol ": = Block" denote "function". "[Programmode: true" is a print control and can be disregarded. "RAd EXPAND: all]" produces $\sqrt{(y^2)} = y$. Line 0 also provides for inputs A, B, C, D, and W. The colon in lines 1 to 19 is the replacement operator " $\leftarrow$ ". Line 1 evaluates $u(x,y) + K1$ where K1 is the integration constant. Line 2 solves $u(x,y) + K1 = 0$ for $y = \omega(x,K1)$. Lines 3 to 6 produce: $B'(y)$, $B'(\omega(x,K1))$, $D(\omega(x,K1))$, and $B(\omega(x,K1))$. Line 7 produces

$$\int \frac{D(\omega(x,K1))\, B'(\omega(x,K1))}{A(x)\, B^3(\omega(x,K1))}\, dx \quad .$$

Lines 8, 9, and 10 produce $R(x,y)$ defined as the right hand side of (10.5). Line 12 is an "administrative" matter. Lines 13-18 solve the differential equation $y' = R(x,y)$ for a given function U. Lines 11 and 20 print out the three results: $u(x,y)$, $R(x,y)$, and the solution to $y' = R(x,y)$ for a given W.

The following are two examples of the application of the algorithm in Table 1.

```
LIE(A,B,C,D,W):= BLOCK (PROGRAMMODE: TRUE, RADEXPAND: ALL].
B1: INTEGRATE (C/A,X)-INTEGRATE (B/D,Y)+K1,
B2: RHS(FIRST(SOLVE(B1,Y))),
B3: DIFF (B,Y,1),
B4: SUBST(B2,Y,B3),
B5: SUBST(B2,Y,D),
B6: SUBST(B2,Y,B),
B7: INTEGRATE(B5*B4/(A*B6↑3),X),
B8: C*D/(A*B)+D/(A*B↑2)/(B7+R),
B9: K1-B1,
G:SUBST(B9,K1,B8),
DISPLAY(B9,G),
W:EV(W,INFEVAL),
F1:SUBST(W,R,G),
M:RATSIMP(-F1/(C*D-F1*A*B)),
N:RATSIMP(1/(C*D-F1*A*B),
F2:INTEGRATE(M,X),
F3:DIFF(F2,Y,1),
F4:INTEGRATE(N-F3,Y),
F5: RATSIMP(F2+F4)
DISPLAY(F5);
```

Table 1. MACSYMA code for obtaining the general first order ODE invariant under the group generated by $U = A(x)B(y)\partial_x + C(x)D(y)\partial_y$ and for solving special cases of the ODE.

Example 1:

$$A = X ,$$

$$B = 1 ,$$

$$C = -1 ,$$

and

$$D = y .$$

Then

$$u(x,y) = \log(xy)$$

and (10.5) becomes

$$\frac{dy}{dx} = \frac{y}{x}\left[-1 + \frac{1}{W(\ln(xy))} \right] . \tag{10.1}$$

If W(x) is selected to be e^x, then the solution to (10.1) is given by the code as

$$y = \frac{\log x - c}{x} .$$

Example 2:

$$A = 1 ,$$

$$B = y$$

and

$$C = D = 1 .$$

Then

$$u(x,y) = \frac{y^2}{2} - x$$

and (10.5) becomes

$$\frac{dy}{dx} = \frac{1}{y} + \frac{1}{y^2}\left(\frac{1}{y} + W(\frac{y^2}{2} - x) \right)^{-1} . \tag{10.2}$$

If W(x) is selected to be W(x) = x, then the solution to (10.2) is given by the code as:

$$y^4 - 4\,x\,y^2 + 8y + 4\,x^2 = c.$$

12. Group Derivable from ODE With Known Solution (Algorithm 2)

The purpose of this Section is to derive one-parameter Lie groups from first order ODE's which have a known solution. These Lie groups have the given ODE as an invariant ODE and will include a larger set of ODE's whose solution can be obtained from the derived Lie group. Thus lists of solved first order ODE's, such as those given in Kamke [8] or Murphy [12], can be used to substantially increase the list of solved first order ODE's. This procedure can be iterated indefinitely. We hope to extend these ideas to higher order ODE's. An advantage to such a program lies in constructing better tables of ODE's and their solutions.

Theorem 9. If

$$y' = w(x,y) \quad , \tag{11.1}$$

with $w(x,y) \in C^{\infty}(\mathcal{O})$, has the one-parameter family of solutions

$$\theta(x,y) = c \tag{11.2}$$

in a neighborhood of $(x_0,y_0) \in \mathcal{O}$, then (11.1) is invariant under the local group with generator

$$U = \partial_x + g(x,y)\partial_y \quad , \tag{11.3}$$

where

$$g(x,y) = e^{\int w_2(x,\psi(x,c))dx} \int w_1(x,\psi(x,c))\, e^{-\int w_2(x,\psi(x,c))dx}\, dx \Bigg|_{c \to \theta(x,y)} \tag{11.4}$$

in which w_1, w_2 denote partial derivatives with respect to the first and second variables and $\psi(x,c)$ is one of the branches of solutions of (11.2) for y.

(Note: The equation $dy/dx = g(x,y)$ may not be solvable in finite terms, in which case, nothing is gained from Theorem 9.)

Proof. From (5.9), $p = w(x,y)$ is invariant under $U = \partial_x + g\partial_y$ if and only if $U'(p-w(x,y)) = 0$ on D: $p = w(x,y)$. Thus $g(x,y)$ must satisfy

$$g_x + wg_y = gw_2 + w_1 \quad . \tag{11.5}$$

From the theory of characteristics for the nonhomogeneous equation (11.5) (see Kamke [7, page 330]), the characteristic equations for (11.5) are

$$\frac{dx}{dt} = 1 \quad , \tag{11.6}$$

$$\frac{dy}{dt} = w \quad , \tag{11.7}$$

and

$$\frac{dg}{dt} = gw_2 + w_1 \quad . \tag{11.8}$$

A solution to (11.6) is $x=t$. (11.7) then becomes

$$\frac{dy}{dt} = w(t,y)$$

with solution

$$\theta(t,y) = c$$

with a branch

$$y = \psi(t,c) \quad .$$

Eq. (11.8) then becomes

$$\frac{dg}{dt} = w_2(t, \psi(t,c))g + w_1(t,\psi(t,c)) \quad , \tag{11.8}$$

which gives (11.4).

13. Examples of Groups Obtained from ODE's With Known Solutions

Example 1. Consider the ODE

$$y' = \frac{y}{x} \tag{12.1}$$

with solution

$$y = cx \ . \tag{12.2}$$

Then (11.4) gives

$$g = \frac{y}{x} \ .$$

Thus the pair (12.1) and (12.2) yield the group

$$U = \partial_x + \frac{y}{x}\,\partial_y \ . \tag{12.3}$$

The group with generator U yields the invariant ODE where W is arbitrary:

$$\frac{dy}{dx} = \frac{y}{x} + x\,W\left(\frac{y}{x}\right) \tag{12.4}$$

which includes (12.1). Eq. (12.4), for any choice of W, yields a solvable equation and the process can be repeated indefinitely.

Example 2. (Linear ODE).

Consider the equation

$$y' = -R(x)y + Q(x) \tag{12.5}$$

with solution

$$y(x) = e^{-\int R(x)dx}\left[\int Q(x)e^{\int R(\bar{x})d\bar{x}}\,dx + c\right] . \tag{12.6}$$

Application of Theorem 9 yields the group G with generator

$$U = \partial_x + \left(-yR(x) + Q(x) + Ce^{-\int R(x)dx}\right)\partial_y \ . \tag{12.7}$$

The first order ODE invariant under the group G is

$$\frac{dy}{dx} + R(x)y - Q(x) = e^{-\int R(x)dx} W\left[y\, e^{\int R(x)dx} - \int Q(x) e^{\int R(x)dx} dx - cx\right] ,$$

where $W \varepsilon C^{\infty}$ is arbitrary.

14. Algorithm for Constructing a Sequence of Integrable ODE's (Algorithm 3)

In this section we mention a possible algorithm by which one could begin with a solvable first order ODE, say

$$d_1: \frac{dy}{dx} = w_1(x,y) \quad , \qquad (13.1)$$

and construct a sequence of solvable first order ODE's:

$$d_i: \frac{dy}{dx} = w_i(x,y) \quad , \qquad (13.i)$$

where d_{i-1} is a special case of d_i. By solvable is meant that the solution to (13.i) can be expressed by a finite number of algebraic and exponential operations and integrations on w_i and on the solutions to (13.j) for $j < i$.

One begins with the generator:

$$U_1 = \partial_x + w_1(x,y)\partial_y \quad .$$

By the method of Section 9 one can obtain an ODE

$$\frac{dy}{dx} = w_2(x,y)$$

which is solvable. One then forms

$$U_2 = \partial_x + w_2(x,y)\partial_y \quad ,$$

and repeats the above procedure with w_2 in place of w_1. This procedure is continued indefinitely.

We hope to present this algorithm in a future paper in greater detail and precision.

ACKNOWLEDGEMENT

The author thanks Roy A. Axford, Garrett Birkhoff, Darryl Holm, Lisa Mantini, and George N. White, Jr., for helpful discussions and for references. We also thank Jo Ann Howell for help in coding.

This work was suggested by a set of lectures Prof. Roy A. Axford gave at the Los Alamos Scientific Laboratory in the summer of 1978.

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